Chapter 8

Legendre Polynomials

8.1 Laplace’s Equation in 3D

Laplace’s Equation appears in many physical situations such as electrostatics, steady-state heat conduction, and the $E = 0$ solution of the Schrödinger equation for a free particle. It is:

$$\nabla^2 u = 0.$$  

If we consider a spherical geometry and use spherical polar coordinates, it can be shown that Laplace’s equation may be written as

$$\nabla^2 u = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2 u}{\partial \phi^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right)$$

where $\theta$ represents the polar angle and $\phi$ represents the azimuthal angle. As should be customary be now, we try the solution

$$u(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi)$$

then substitute and divide by $u(r, \theta, \phi)$

$$\nabla^2 u = \frac{\sin^2(\theta)}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} + \frac{\sin \theta}{\Theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = 0.$$  

We see immediately that the $\Phi$ term is independent and thus begin the game of setting sections equal to constants. Because we want the $\Phi$ part of the solution to oscillate, we set

$$\frac{d^2 \Phi}{d\phi^2} = -m^2 \Phi,$$

a differential equation that has solutions

$$\Phi = \left\{ e^{im\phi}, e^{-im\phi} \right\}.$$
In other words, we have chosen \( m^2 \) to be negative so \( \Phi \) comes out periodic for integer \( m \). The requirement that \( u(r, \theta, \phi) \) be single valued, and so \( \Phi(\phi + 2\pi) = \Phi(\phi) \) then quantizes the modes to have integer values of \( m \) in regard to their azimuthal dependence.

Moving on to the radial and polar angle portions of the solution we can arrange things so that we separate the variables there too. We have:

\[
\frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = \frac{m^2}{\sin^2(\theta)} - \frac{1}{\sin \theta \Theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = \ell(\ell + 1).
\]

The reasoning for the chosen constant will become clear momentarily. The radial equation may now be written as

\[
\frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - \ell(\ell + 1) = 0.
\]

While the polar angle dependence is expressed as

\[
\frac{1}{\sin \theta \Theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \ell(\ell + 1) - \frac{m^2}{\sin^2(\theta)} = 0.
\]

We next change variables to \( x = \cos \theta \) and \( y(x) = \Theta(\theta) \). The equation that governs the function \( \Theta(\theta) \) is then recast according to the changes

\[
\frac{dy}{dx} = \frac{d\Theta}{dx} = \frac{d\Theta}{d\theta} \frac{dx}{d\theta} = -\frac{d\Theta}{d\theta} \frac{1}{\sin \theta}
\]

thereby becoming

\[
\frac{d}{dx} \left( [1 - x^2] \frac{dy}{dx} \right) + \ell(\ell + 1)y(x) - \frac{m^2}{1 - x^2} y(x) = 0.
\]

When \( m = 0 \), this is a version of Legendre’s Equation and in the case of \( m \neq 0 \) the solutions are the associated Legendre polynomials. In general, \( \ell \) is also required to be an integer (for further explanation see below). The full angular dependence of normal modes is

\[
P_\ell^m(\cos \theta) e^{\pm im\phi},
\]

where \( P_\ell^m(\cos \theta) \) is an associated Legendre polynomial that, for \( m = 0 \), becomes the Legendre polynomial we will spend much of the rest of the chapter exploring. These products of \( \Theta(\theta) \) and \( \Phi(\phi) \) appear often enough that they are, up to a constant, defined as the spherical harmonics that may be familiar to your from quantum mechanics problems.
We note that, if the right-hand side of Laplace’s equation is zero, then the radial part of the normal modes is governed by:

\[ r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} - \ell(\ell + 1)R(r) = 0. \]  \hspace{1cm} (8.1)

The solution to this equation is a linear combination of \( r^\ell \) and \( r^{-\ell-1} \). If \( r = 0 \) is part of the solution domain then only the solution that is regular as \( r \to 0 \) can be present and we have \( R_\ell(r) = r^\ell \). If we need a solution that is bounded as \( r \to \infty \) then only \( r^{-\ell-1} \) is allowed. Whatever the boundary conditions the normal modes are then:

\[ R_\ell(r)P^m_\ell(\cos \theta)e^{im\phi}, \]

where \( m \) can range from \(-\ell\) to \( \ell \). This requirement is due to the fact that the associated Legendre polynomials are zero if \(|m| > |\ell|\). This relates to the quantum mechanical \( z \)-component of the angular momentum having to obey \(|m| < |\ell|\). The general solution is then

\[ u(r, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} C_{\ell m}R_\ell(r)P^m_\ell(\cos \theta)e^{im\phi}. \]

### 8.2 The return of Legendre polynomials

When solving the Laplace Equation, \( \nabla^2 u = 0 \), in spherical geometry, we encounter the equation

\[ (1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0, \]  \hspace{1cm} (8.2)

where, the \( \theta \)-dependent part of the normal modes is related to the solution of this equation by:

\[ \Theta(\theta) = y(\cos \theta), \]

provided that \( m = 0 \), i.e., the normal mode is independent of \( \phi \) (azimuthal symmetry). We are interested in solving (8.2) for \(-1 \leq x \leq 1\), i.e., \( \theta \) between 0 and \( \pi \). Meanwhile, \( n \) is the separation constant (denoted \( \ell \) above). Once you specify \( n \) and \( m = 0 \), the normal modes for the solution of Laplace’s equation are:

\[ u_n(r, \theta, \phi) = (a_n r^n + b_n r^{-n-1}) y_n(\cos \theta). \]

Now recall that we have seen Eq. (8.2) before. This is the equation we solved by series back in the Chapter on Frobenius’ method for solving linear ordinary differential equations. We recall that, provided \( n \) is an integer, Eq. (8.2) has one terminating solution and one solution that diverges at \( \cos \theta = \pm 1 \). Here, we choose \( y_n(\cos \theta) \) to be the terminating solution of (8.2). Otherwise our normal modes diverge at \( \cos \theta = \pm 1 \). Remember that the terminating solution was \( y_1(x) \) for \( n \) even and \( y_2(x) \) for \( n \) odd.
These terminating solutions, known as Legendre polynomials, can be summarized in the formula

\[ P_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k(2n - 2k)!}{2^k k!(n - k)!(n - 2k)!} x^{n-2k}. \]  

(8.3)

These satisfy (8.2) as polynomials of degree \( n \). Meanwhile, the upper limit of the sum is the largest integer less than or equal to \( n/2 \):

\[ \left[\frac{n}{2}\right] = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd}. \end{cases} \]

We also notice that \( P_n(x) \) is an even function of \( x \) if \( n \) is even and an odd function of \( x \) if \( n \) is odd. Working out the first few:

\[ P_0(x) = 1; \quad P_1(x) = x; \quad P_2(x) = \frac{3}{2} x^2 - \frac{1}{2}. \]  

(8.4)

The general \( \phi \)-independent solution to the Laplace equation that is finite at \( \theta = 0, \pi \) is then:

\[ u(r, \theta, \phi) = \sum_{n=0}^{\infty} (a_n r^n + b_n r^{-n-1}) P_n(\cos \theta). \]

8.2.1 Rodrigues’ formula

We can recast Eq. (8.3) as:

\[ P_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k}{2^k k!(n - k)!} \left( \frac{d}{dx} \right)^n x^{2n-2k} \]

\[ = \frac{1}{2^n n!} \left( \frac{d}{dx} \right)^n \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k n!}{k!(n - k)!} x^{2n-2k}. \]

We can extend the sum to \( n \) because the extra terms we add by doing that give 0 after differentiation. The sum is then the binomial theorem for \( (x^2 - 1)^n \), so we have:

\[ P_n(x) = \frac{1}{2^n n!} \left( \frac{d}{dx} \right)^n (x^2 - 1)^n. \]  

(8.5)

This is known as Rodrigues’ formula and is an easier way to compute Legendre polynomials in many cases.
8.3 Generating Function

Provided that $|t| < 1$, the generating equation for Legendre polynomials can be written as

$$g(t, x) = \sum_{n=0}^{\infty} P_n(x) t^n = \frac{1}{\sqrt{1 - 2xt + t^2}}. \quad (8.6)$$

This means that $P_n(x)$ can be found by computing the $n$th derivative of this function with respect to $t$ and setting $t = 0$. Eq. (8.6) is, after all, a Taylor series in $t$, with coefficients $P_n(x)$:

$$P_n(x) = \frac{1}{n!} \left. \frac{\partial^n g}{\partial t^n} \right|_{t=0}.$$}

The Legendre polynomials can then be extracted from the generating function by defining $y = 2xt - t^2$ and expanding in powers of $y$:

$$g(y) = (1 - y)^{-\frac{1}{2}} = 1 + \frac{y}{2} + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!} y^2 + \cdots .$$

We next wish to prove (8.6). The function $g$ obeys the equation

$$(1 - x^2) \frac{\partial^2 g}{\partial x^2} - 2x \frac{\partial g}{\partial x} + t \frac{\partial^2}{\partial t^2} (tg) = 0.$$}

Let’s write

$$g(t, x) = \sum_{n=0}^{\infty} \xi_n(x) t^n,$$

where the functions $\xi_n(x)$ are, as yet, undetermined. Substituting, we find

$$(1 - x^2) \sum_{n=0}^{\infty} \xi_n''(x) t^n - 2x \sum_{n=0}^{\infty} \xi_n'(x) t^n + \sum_{n=0}^{\infty} (n+1)(n) \xi_n(x) t^n = 0.$$}

Now, because the power series in $t$ is unique, we must have

$$(1 - x^2) \frac{d^2 \xi}{dx^2} - 2x \frac{d \xi}{dx} + (n+1)(n) \xi = 0,$$

so $\xi$ satisfies the Legendre equation. But what is $\xi_n(1)$?

$$g(t, 1) = \sum_{n=0}^{\infty} \xi_n(1) t^n = \frac{1}{1 - t}.$$}

But we can expand the right-hand side as a power series using the geometric series:

$$\frac{1}{1 - t} = \sum_{n=0}^{\infty} t^n.$$
Therefore, by the uniqueness of power series, \( \xi_n(1) = 1 \). So suppose now we have

\[
\xi_n(x) = \alpha P_n(x) + \beta Q_n(x),
\]

i.e., we admit the possibility that \( \xi_n(x) \) is a linear combination of the terminating solution and the solution that does not terminate. But if \( \xi_n(1) = 1, \beta = 0 \) since \( Q_n \) diverges at \( x = 1 \) and so \( \alpha = 1 \) \( (P_n(1) = 1) \).

This also provides us with a way to think about Legendre polynomials: they are functions that (1) satisfy Legendre’s equation (2) are polynomials (I know, shocking) and (3) = 1 at 1.

Let’s use the generating function to calculate \( P_n(-1) \).

\[
g(t, -1) = \sum_{n=0}^{\infty} P_n(-1)t^n = \frac{1}{\sqrt{1 + 2t + t^2}} = \frac{1}{1 + t} = \sum_{n=0}^{\infty} (-1)^n t^n.
\]

So

\[
P(-1) = (-1)^n.
\]

By a generalization of this argument we can also use the generating function to prove the statement made above, that \( P_n(x) \) is even if \( n \) is even and odd if \( n \) is odd.

### 8.4 Derivation of two recurrence relations from the generating function

In this section we use the generating function to derive two recurrence relations for Legendre polynomials. These relations are useful and their derivation using the recurrence relation is a useful exercise in manipulating series, but none of the material in this section is essential. The recurrence relations obtained are often the best way to generate the next Legendre polynomial if you have two, i.e., you can take \( P_0(x) \) and \( P_1(x) \) and use them to generate \( P_2(x) \) then use \( P_1 \) and \( P_2 \) to generate \( P_3 \), etc.

The first derivation begins by observing that

\[
\frac{\partial g}{\partial x} = \frac{t}{(1 - 2tx + t^2)^{3/2}} = \frac{t}{(1 - 2tx + t^2)^2}g
\]

and so

\[
(1 - 2tx + t^2)\frac{\partial g}{\partial x} - tg = 0.
\]

But,

\[
\Rightarrow (1 - 2tx + t^2) \sum_{n=0}^{\infty} P'_n(x)t^n - t \sum_{n=0}^{\infty} P_n(x)t^n = 0
\]

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1X_n=0 P_0(x) t^n + \sum_{n=0}^{\infty} P'_n(x) t^{n+2} - 2 \sum_{n=0}^{\infty} P'_n(x) t^{n+1} x - \sum_{n=0}^{\infty} P_n(x) t^{n+1}.

We find, from the “indicial equation” (lowest power of $t$ here):

\[ P'_0(x) = 0, \]

which is obviously true with the knowledge that $P_0(x) = 1$. Continuing,

\[ \sum_{n=0}^{\infty} P'_n(x) t^{n+2} + \sum_{n=0}^{\infty} P'_{n+1}(x) t^{n+1} - 2P'_n(x) t^{n+1} x - P_n(x) t^{n+1} = 0 \]

For the $t^1$ term,

\[ P'_1(x) - 2xP'_0(x) - P_0(x) = 0 \]

This statement is once again true with the knowledge that $P_0 = 1$ and $P_1 = x$. We once more shift dummy variables with the removal of the terms for which we have already accounted.

\[ \sum_{n=0}^{\infty} \left[ P'_{n+2}(x) - 2xP'_{n+1}(x) + P'_n(x) - P_{n+1}(x) \right] t^{n+2} = 0 \]

Forcing the bracket expression $= 0$ defines the desired recursion relation. We will write it with the index decreased by one to suggest the usual convention of finding the $n + 1^{st}$ polynomial from previous polynomials.

\[ P'_{n+1}(x) + P'_{n-1}(x) = 2xP'_n(x) + P_n(x). \]

To get the second recursion relation we begin by computing

\[ \frac{\partial g}{\partial t} = \frac{x-t}{(1 - 2tx + t^2)^3} = \sum_{n=1}^{\infty} nP_n(x)t^{n-1} = \frac{x-t}{(1 - 2tx + t^2)^3} g \]

\[ \Rightarrow (1 - 2tx + t^2) \sum_{n=1}^{\infty} nP_n(x)t^{n-1} + (t - x) \sum_{n=0}^{\infty} P_n(x)t^n = 0 \]

\[ \sum_{n=1}^{\infty} nP_n(x)t^{n-1} - 2x \sum_{n=1}^{\infty} nP_n(x)t^n + \sum_{n=1}^{\infty} nP_n(x)t^{n+1} \]

\[ + x \sum_{n=0}^{\infty} P_n(x)t^n - \sum_{n=0}^{\infty} P_n(x)t^{n+1} = 0 \quad (8.7) \]

Extracting the coefficient of the $t^0$ term,

\[ P_1 - xP_0 = 0. \]
Once again, this is a true statement. We next pull out coefficients of $t^1$

$$2P_2(x) - 2xP_1(x) - xP_1(x) + P_0(x) = 0$$

Using the known value of $P_2(x) = \frac{1}{2} (3x^2 - 1)$, this statement is also found to be true. We next rewrite the LHS under one summation

$$\sum_{n=0}^{\infty} [(n + 3)P_{n+3}(x) - 2x(n + 2)P_{n+2}(x) + (n + 1)P_{n+1}(x)$$

$$- xP_{n+2}(x) + P_{n+1}(x)]t^{n+2} \quad (8.8)$$

We may extract the terms within the brackets and decrement the indices by 2 to produce a recursion relation.

$$(n + 1)P_{n+1}(x) - 2x(n)P_n(x) + (n - 1)P_{n-1}(x) - xP_n(x) + P_{n-1}(x) = 0.$$ 

Cleaning up the expression results in

$$(n + 1)P_{n+1}(x) + nP_{n-1}(x) = (2n + 1)xP_n(x).$$

### 8.5 Why the magic happens: Multipole Expansion

In this section we examine the physics content of the generating function. It will turn out that the reason the generating function looks the way it does is because there are two equivalent ways to write the electrostatic potential of a point charge located at a point on the $z$-axis.

The electric potential in spherical coordinates due to a point charge located on the $z$-axis at $z = a$ can be written as

$$\varphi(\vec{r}) = \frac{q}{4\pi \epsilon_0 ||\vec{r} - \vec{a}||}$$

$$||\vec{r} - \vec{a}||^2 = (\vec{r} - \vec{a}) \cdot (\vec{r} - \vec{a}) = r^2 + a^2 - 2ar \cos \theta$$

$$\varphi(\vec{r}) = \frac{q}{4\pi \epsilon_0 \sqrt{r^2 + a^2 - 2ar \cos \theta}}$$

$$= \frac{q}{4\pi \epsilon_0 r} \frac{1}{\sqrt{1 - \frac{2a}{r} \cos \theta + \frac{a^2}{r^2}}}$$

This is noticed to be the Legendre polynomial generating function for $x = \cos \theta$ and $t = \frac{a}{r}$. Thus, provided that $|\frac{a}{r}| < 1$, the potential may be written as

$$\varphi(\vec{r}) = \frac{q}{4\pi \epsilon_0 r} \sum_{n=0}^{\infty} P_n(\cos \theta) \left(\frac{a}{r}\right)^n. \quad (8.9)$$
We now give an alternative, more physics-y, derivation of this equation. Because the charge is on the $z$-axis, we would expect that the potential is azimuthally symmetric, i.e., independent of $\phi$. Eq. (8.9) emphasizes that the potential can be written as a sum of normal modes. The $r$-dependence is forced to be $1/r^{n+1}$ in each normal mode, because we are interested in the region where $r > a$, and so we need the potential to be finite as $r \to \infty$. The coefficient of the $n$th term is then $a^n$. This can be seen if we just consider the potential along the $z$-axis, where it takes the form

$$\varphi(r = r\hat{z}) = \frac{1}{4\pi\varepsilon_0} \frac{1}{r - a} = \frac{1}{4\pi\varepsilon_0} \frac{1}{r} \left( 1 - \frac{a}{r} \right)$$

for $r > a > 0$.

Whether we arrive at Eq. (8.9) by the generating function or by the physics argument based on the normal modes, in either case we have

$$\varphi = \frac{q}{4\pi\varepsilon_0 r} \left[ 1 + P_1(\cos \theta) \frac{a}{r} + \mathcal{O}\left( \frac{a}{r} \right)^2 \right],$$

which gets us used to the idea that the expansion (8.9) is useful if we are interested in describing the potential for $r \gg a$.

Alternatively, if $r < a$, we wish to pull out a factor of $a$ from the denominator to allow the value of the variable $t$ in our generating function definition to remain less than 1. The general result may be stated as

$$\frac{1}{||\vec{r}_1 - \vec{r}_2||} = \sum P_n(\cos \theta) \left( \frac{r_<}{r_>} \right)^n \frac{1}{r_>},$$

with

$$r_< = \begin{cases} r_1 & \text{if } r_1 < r_2 \\ r_2 & \text{if } r_1 > r_2, \end{cases}$$

and $r_>$ the bigger of the two.

### 8.5.1 Example: The Electric Dipole

A positive and negative charge are situated such that the positive charge $+q$ is located along the positive $z$-axis at a distance $a$ from the origin while the negative charge $-q$ is located along the negative $z$-axis at a distance $a$ from the origin. We would like to calculate the electric potential at a point $\vec{r}$ from the origin. We establish $r_1$ and $r_2$ as the vectors connecting this point with the positive and negative charges respectively

$$\varphi(\vec{r}) = \frac{q}{4\pi\varepsilon_0} \left( \frac{1}{r_1} - \frac{1}{r_2} \right);$$

$$r_1 = \vec{r} - a\hat{z};$$

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\[ r_2 = \vec{r} + a\hat{z}. \]

So

\[
\varphi(\vec{r}) = \frac{q}{4\pi\varepsilon_0} \left( \frac{1}{\sqrt{r^2 + a^2 - 2ar \cos \theta}} - \frac{1}{\sqrt{r^2 + a^2 + 2ar \cos \theta}} \right).
\]

If we take the case that the magnitude of \( r \) is greater than the magnitude of \( a \),

\[
\phi(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r} \left( \frac{1}{\sqrt{1 - \frac{2a}{r} \cos \theta + \frac{a^2}{r^2}}} - \frac{1}{\sqrt{1 + \frac{2a}{r} \cos \theta + \frac{a^2}{r^2}}} \right)
\]

\[
= \frac{q}{4\pi\varepsilon_0 r} \left[ \sum_{n=0}^{\infty} P_n(\cos \theta) \left( \frac{a}{r} \right)^n - \sum_{n=0}^{\infty} P_n(\cos \theta) \left( -\frac{a}{r} \right)^n \right]
\]

\[
= \frac{q}{4\pi\varepsilon_0 r} \sum_{n=0}^{\infty} (1 - (-1)^n) P_n(\cos \theta) \left( \frac{a}{r} \right)^n
\]

\[
= \frac{2q}{4\pi\varepsilon_0 r} \sum_{n=0}^{\infty} P_{2n+1}(\cos \theta) \left( \frac{a}{r} \right)^{2n+1}
\]

\[
= \frac{2q}{4\pi\varepsilon_0 r} \left( \left( \frac{a}{r} \right) \cos \theta + \left( \frac{a}{r} \right)^3 \left( \frac{5}{2} \cos^3 \theta - \frac{3}{2} \cos \theta \right) + \cdots \right).
\]

So, finally:

\[
\varphi(\vec{r}) = \frac{2qa}{4\pi\varepsilon_0 r^2} \cos \theta + \mathcal{O} \left( \frac{qa^3}{4\pi\varepsilon_0 r^4} \right).
\]

In the limit \( a \ll r \), the effects of the truncation of the expression at the level of \( (\frac{a}{r})^3 \) are minimal. The results of this example give insight into how one could tell that you are looking at a dipole as opposed to a point charge at the origin even from large distances. The dipole yields a potential which is not spherically symmetric. This potential falls off as \( 1/r^2 \) at large distances, and its strength is given by the “dipole moment” \( 2qa \).

### 8.6 Orthogonality of the \( P_n \)’s

**Claim:**

\[
\int_{-1}^{1} P_n(x)P_m(x) \, dx = 0 \quad \text{if} \quad n \neq m.
\]

**Proof:**

Multiply the D.E. for \( P_n(x) \) by \( P_m(x) \)

\[
P_m(x) \frac{d}{dx} \left[ (1-x^2) \frac{dP_n(x)}{dx} \right] + P_m(x)n(n+1)P_n(x) = 0
\]
If we integrate from -1 to 1 we get

\[ \int_{-1}^{1} P_m(x) \frac{d}{dx} \left[(1-x^2) \frac{dP_n(x)}{dx}\right] dx + n(n+1) \int_{-1}^{1} P_m(x) P_n(x) dx = 0 \]

\[ \Rightarrow P_m(x) \left[(1-x^2) \frac{dP_n(x)}{dx}\right]_{-1}^{1} - \int_{-1}^{1} \frac{dP_m(x)}{dx} \left[(1-x^2) \frac{dP_n(x)}{dx}\right] dx \]

\[ + n(n+1) \int_{-1}^{1} P_m(x) P_n(x) dx = 0 \quad (8.11) \]

Because the first term is zero always we find the equality of two integrals

\[ \int_{-1}^{1} \frac{dP_m(x)}{dx} \left[(1-x^2) \frac{dP_n(x)}{dx}\right] dx = n(n+1) \int_{-1}^{1} P_m(x) P_n(x) dx \]

Now we interchange the roles of \( n \) and \( m \) and repeat the argument. We obtain:

\[ \int_{-1}^{1} \frac{dP_n(x)}{dx} \left[(1-x^2) \frac{dP_m(x)}{dx}\right] dx = m(m+1) \int_{-1}^{1} P_m(x) P_n(x) dx \]

But the integral on the left-hand side is same on both sides. So, by the definition of equality,

\[ n(n+1) \int_{-1}^{1} P_m(x) P_n(x) dx = m(m+1) \int_{-1}^{1} P_m(x) P_n(x) dx \]

\[ \Rightarrow [n(n+1) - m(m+1)] \int_{-1}^{1} P_m(x) P_n(x) dx = 0. \]

There are two ways to satisfy this equation

1. \( n = m \)
2. \( \int_{-1}^{1} P_m(x) P_n(x) dx = 0 \)

and thus the original claim is proven.

## 8.7 Legendre Series Expansion

We now consider a function, \( f(x) \) defined for \(-1 \leq x \leq 1\) and we will expand \( f(x) = \sum_{n=0}^{\infty} C_n P_n(x) \). This is called a Legendre series. To calculate the \( C_n \)'s, we write

\[ \int_{-1}^{1} f(x) P_n(x) dx = \sum_{n=0}^{\infty} \left[ \int_{-1}^{1} P_m(x) P_n(x) dx \right] C_n = \left[ \int_{-1}^{1} P_m^2(x) \right] C_m. \quad (8.12) \]
Let’s calculate the integral using the generating function:

\[ \frac{1}{1 - 2xt + t^2} = \left[ \sum_{n=0}^{\infty} P_n(x)t^n \right]^2 = \sum_{n,m=0}^{\infty} P_n(x)P_m(x)t^{n+m}. \]

When we integrate -1 to 1, the cross terms go away

\[ \int_{-1}^{1} \frac{dx}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} \left[ \int_{-1}^{1} P_n^2(x)dx \right] t^{2n}. \]

Because we wish to exploit the property of uniqueness of power series, we write the power series for the LHS.

\[ y = 1 - 2xt + t^2 \quad dy = -2tdx \]

\[ \int_{-1}^{1} \frac{dx}{1 - 2xt + t^2} = -\frac{1}{2t} \int_{(1-t)^2}^{(1+t)^2} \frac{dy}{y} = -\frac{1}{2t} \left[ \log \frac{(1-t)^2}{(1+t)^2} \right]. \]

To summarize,

\[ \frac{1}{t} \left[ \log \frac{(1+t)}{(1-t)} \right] = \sum_{n=0}^{\infty} \left[ \int_{-1}^{1} P_n^2(x)dx \right] t^{2n}. \]

We next expand the LHS into a power series using the fact \( \log \frac{1+t}{1-t} = \log (1+t) - \log (1-t) \) where \( \log (1-t) = -\sum_{n=1}^{\infty} \frac{t^n}{n} \) for \( |t| < 1 \). This leads us to the identity

\[ 2 \sum_{n=0}^{\infty} \frac{t^{2n}}{2n+1} = \sum_{n=0}^{\infty} \left[ \int_{-1}^{1} P_n^2(x)dx \right] t^{2n} \]

Then, by the uniqueness of power series,

\[ \int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n+1}. \]

This, then, is the integral we would get if we consider the orthogonality equation [8.10] for \( n = m \), i.e. for arbitrary \( n \) and \( m \) we can say

\[ \int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1} \delta_{nm}. \quad (8.13) \]

Returning to equation [8.12], it follows that

\[ C_n = \frac{2n+1}{2} \int_{-1}^{1} f(x)P_n(x)dx. \]
8.7.1 The Earth

We know by now that potentials tend to be Legendre series:

\[ \varphi(r, \theta) = \sum_{n=0}^{\infty} \left( A_n r^n + B_n r^{-n-1} \right) P_n(\cos \theta). \]

If we consider the gravitational potential, \( u(r, \theta) \) of the Earth, then

\[ u(r, \theta) = \sum_{n=0}^{\infty} B_n \frac{1}{r^{n+1}} P_n(\cos \theta), \]

where the coefficient \( A_n \) is zero based on the requirement \( u(r \to \infty, \theta) = 0 \). We know that \( B_0 = -GM \) so

\[ u(r, \theta) = -\frac{GM}{r} + \sum_{\ell=1}^{\infty} B_\ell \frac{P_\ell(\cos \theta)}{r^{\ell+1}}. \]

Let’s write

\[ B_\ell = GM R^\ell a_\ell, \]

where \( a_\ell \) is dimensionless and \( R \) is the radius of the Earth. Then

\[ u(r, \theta) = -\frac{GM}{R} \left[ \frac{R}{r} - \sum_{\ell=2}^{\infty} a_\ell \left( \frac{R}{r} \right)^{\ell+1} P_\ell(\cos \theta) \right]. \]

Note:

\[ \frac{GM R^\ell a_\ell}{r^{\ell+1}} = \frac{2\ell + 1}{2} \int_{-1}^{1} d(\cos \theta) u(r, \theta) P_\ell(\cos \theta). \]

Research into the field has determined

\[ a_2 = (1,082,635 \pm 11) \times 10^{-9} \]
\[ a_3 = (-2,531 \pm 7) \times 10^{-9} \]
\[ a_4 = (-1,600 \pm 12) \times 10^{-9}. \]

Thus the Earth is not a perfect sphere, but it is one to one part in a thousand, and its gravitational potential can be summarized with excellent accuracy by the \( \ell = 0 \) (monopole) term and the term with \( \ell = 2 \) (quadrupole). Note that the \( \ell = 1 \) term vanishes because the origin from which the potential is measured is chosen to be the Earth’s center of mass, and so \( \int \vec{r}dm = 0 \).
8.8 Spherical Harmonics

The solution to the separation of variables problem in $\theta$ and $\phi$ associated with separation constants $\ell$ and $m$ can be written as

$$Y_{\ell m}(\theta, \varphi) = (-1)^m \sqrt{(2\ell + 1)(\ell - m)!} \frac{\ell^{m\varphi} P^m_\ell(\cos \theta)}{4\pi(\ell + m)!}.$$ 

As a reminder, the validity of this solution still requires $|m| \leq \ell$. The general solution of Laplace’s Equation is then

$$\Psi(r, \theta, \varphi) = \sum_{\ell=0,|m|\leq\ell} \infty (a_{\ell r^\ell} + b_{\ell r^{\ell-1}})Y_{\ell m}(\theta, \varphi).$$

When $m = 0$, these solutions are proportional to the Legendre polynomials (albeit with a different normalization).

A few properties of these Spherical harmonics are

$$\int_0^\pi \int_0^{2\pi} d\Omega Y_{\ell m}(\theta, \varphi) Y_{\ell' m'}^*(\theta, \varphi) = \delta_{\ell \ell'} \delta_{mm'}$$

$$Y_{\ell m}^*(\theta, \varphi) = (-1)^m Y_{\ell,-m}(\theta, \varphi).$$

We know the addition theorem

$$\cos \gamma = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos (\varphi_1 - \varphi_2).$$

This can be written in a fancy way as:

$$P_1(\cos \gamma) = \frac{4\pi}{3} \sum_{m=-1}^{\ell} Y_{\ell m}(\theta_1, \varphi_1)Y_{\ell m}^*(\theta_2, \varphi_2)$$

and then generalizes to

$$P_n(\cos \gamma) = \frac{4\pi}{2n + 1} \sum_{m=-n}^{\ell} Y_{nm}(\theta_1, \varphi_1)Y_{nm}^*(\theta_2, \varphi_2).$$

8.8.1 Ring of Charge

Consider a ring in the $x - y$ plane with uniformly distributed charge totaling $q$. We wish to calculate $\Psi(r, \theta, \phi) \rightarrow \Psi(r, \theta)$ with azimuthal symmetry. We will show that, if the potential is easy to calculate along a particular line, you may use the knowledge of $\Psi$ there to obtain the potential which satisfies the Laplace equation everywhere except on the ring. This works because

$$\nabla^2 \Psi(r, \theta) = 0$$
and so
\[ \Psi(r, \theta) = \sum_{L=0}^{\infty} \left( a_L r^L + b_L r^{-(L+1)} \right) P_L(\cos \theta). \]

We next impose the customary boundary condition for potentials
\[ \Psi(r \to \infty, \theta) = 0 \]
which requires \( a_L = 0 \) for all \( L \). For \( r > a \), we can then write
\[ \Psi(r, \theta) = \sum_{L=0}^{\infty} \frac{b_L}{r^{L+1}} P_L(\cos \theta) \equiv \frac{q}{4\pi \varepsilon_0 r} \sum_{L=0}^{\infty} \frac{a^{L}c_L}{r^{L}} P_L(\cos \theta). \quad (8.14) \]

In this way, the problem has been reduced to calculating the (dimensionless) \( c_L \)'s.

Along the \( z \)-axis, we know
\[ \Psi(r, 0) = \int \frac{dq}{4\pi \varepsilon_0 \sqrt{a^2 + z^2}} = \frac{q}{4\pi \varepsilon_0 \sqrt{a^2 + z^2}}. \]

We will use this knowledge to write an expression for the \( b_L \)'s. Note that, at this point, the physics is technically done. The rest will be mathematical trickery (hopefully not the deceptive kind) to write the above expression into a series of Legendre polynomials (the easiest kind with \( \theta = 0 \) and the fact that \( P_L(1) = 1 \)), which can then be equated with equation (8.14) at the point \( \theta = 0 \). Once the power series in \( r \) has determined certain choices for the \( b_L \)'s in order to obtain the correct dependence on \( r \) along the line \( \theta = 0 \) those coefficients then determine the linear combination of normal modes that we get at an arbitrary value of \( \theta \).

We carry out our cunning plan by rewriting our expression for the potential along the \( z \)-axis as:
\[ \Psi(r, 0) = \frac{q}{4\pi \varepsilon_0 r} \left( 1 + \frac{a^2}{r^2} \right)^{-\frac{1}{2}} = \frac{q}{4\pi \varepsilon_0 r} \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right) \left( \frac{a^2}{r^2} \right)^n, \]
and then observing that
\[ \left( -\frac{1}{2} \right)^n = \frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{1}{2}-(n-1))}{n!} = \frac{(-1)^n(1\cdot3\cdots(2n-1))}{2^nn!}, \]
and we must define the term to be 1 when \( n = 0 \). The conversions here become easier to see when it is noted that the number of terms in the numerator will be equal to \( n \) for any \( n \neq 0 \). Combining these expressions with the added definition for \( n = 0 \) we obtain
\[ \left( -\frac{1}{2} \right)^n = \frac{(-1)^n(2n-1)!!}{2n!!}. \]
To summarize, if $\theta = 0$,

$$\Psi(r, 0) = \frac{q}{4\pi \varepsilon_0 r} \sum_{n=0}^{\infty} \frac{(-1)^n (2n - 1)!!}{2n!!} \left( \frac{a^2}{r^2} \right)^n = \frac{q}{4\pi \varepsilon_0} \sum_{n=0}^{\infty} \frac{(-1)^n (2n - 1)!!}{2n!!} \left( \frac{a^{2n}}{r^{2n+1}} \right).$$

We make this look like equation (8.14) by writing

$$c_L = \begin{cases} 0 & \text{if } L \text{ is odd} \\ \frac{(-1)^n (2n - 1)!!}{2n!!} & \text{if } L = 2n \text{ is even}. \end{cases}$$

But this determines the coefficients in Eq. (8.14) for $\theta = 0$ and once they are determined there they are the same everywhere (they are, after all constants, i.e., the same for all $r$ and $\theta$).

So for any $r > a$ and any $\theta$, we are now able to write

$$\Psi(r, \theta) = \sum_{n=0}^{\infty} \frac{(-1)^n (2n - 1)!!}{2n!!} \frac{q}{4\pi \varepsilon_0 r^{2n+1}} a^{2n} P_{2n}(\cos \theta). \quad (8.15)$$

By using the solution along the $z$-axis and the fact that we know the general form of the solution of Laplace’s equation we are able to calculate the general form of this potential.